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FREE CONVECTION BOUNDARY LAYERS IN A SATURATED POROUS MEDIUM WITH LATERAL MASS FLUX

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Abstract – The effects of uniform lateral mass flux on the free convection boundary layer on a vertical wall in a saturated porous medium are considered. A series valid near the leading edge is derived and this is extended by a numerical solution of the full equations. Asymptotic expansions, valid at large distances along the plate, are derived in both the cases of withdrawal and injection of fluid. In the former case the boundary layer has constant thickness, while in the latter case there is a region of constant temperature next to the wall made up of fluid that has been injected through the wall, with an outer region where thermal diffusion is important.

NOMENCLATURE

g,	acceleration due to gravity;
K,	permeability of the porous medium;
Q,	heat-transfer coefficient;
T,	temperature;
T_w,	wall temperature;
T_0,	temperature of the ambient fluid;
ΔT,	temperature difference $= \lvert T_w - T_0 \rvert$;
u,	Darcy's law velocity in the x-direction;
v,	Darcy's law velocity in the y-direction;
V_w,	transpiration velocity (constant);
x,	vertical coordinate;
y,	horizontal coordinate.

Greek symbols

α,	equivalent thermal diffusivity;
β,	coefficient of thermal expansion;
μ,	viscosity of the convective fluid;
ρ_0,	density of the convective fluid.

1. INTRODUCTION

IN A RECENT paper, Cheng and Minkowycz [1] discussed the free convection boundary layer on an impermeable vertical wall embedded in a saturated porous medium. This problem arose in modelling the cooling of a hot intrusive trapped in an aquifer in which the groundwater next to the intrusive is not vaporised. Cheng [2] has also considered the effects that the lateral injection or withdrawal of fluid through the wall has on the free convection boundary layer. He applied this to the problems of the convective movement of water discharged from a geothermal power plant into groundwater of a different temperature and in the natural recharging of an aquifer by groundwater of a different temperature. A full description of the underlying physical assumptions is given in [1] and [2] and it is unnecessary to repeat the details here.

Cheng [2] considered the case of large Rayleigh number where the governing equations are the boundary-layer equations as derived by Wooding [3]. To solve these, Cheng [2] looked for those power law variations of wall temperature and transpiration velocity for which a similarity solution could be obtained. He found that this was possible when the wall temperature and transpiration velocity varied as x^{λ} and $x^{(\lambda-1)/2}$ respectively. Though giving a good insight into the nature of the problem, this method has the disadvantage that the boundary conditions that are necessarily imposed on the wall temperature and transpiration velocity are unrealistic.

In this paper we consider the effects of the lateral injection or withdrawal at constant velocity V_w of fluid at constant temperature T_w on the free convection boundary layer on a vertical plane wall in a saturated porous medium with ambient temperature T_0. The analogous problem of a free convection boundary layer on a vertical plate with constant blowing or suction has been treated by Merkin [4] and methods similar to those given in [4] are used to solve the present problem.

A series expansion in powers of $x^{1/2}$ is first obtained which describes the flow near the leading edge. This expansion is then extended by a numerical solution of the boundary-layer equations, which starts at $x = 0$ and proceeds along the wall until the asymptotic conditions are attained to the required accuracy. Asymptotic expansions (i.e. for large x) are then derived in both the cases of withdrawal and injection of fluid. In the former case the boundary layer is found to have constant thickness and that this is approached through terms which are exponentially small for large x. In the latter case the flow is divided up into two regions. There is a region of thickness $O(x)$ next to the wall made up of fluid at temperature T_w which has been injected through the wall, and in which thermal diffusion can be neglected. The edge of this inner region is determined by the streamline that emerged from the wall at the leading edge, and this solution has a discontinuity in temperature at this dividing streamline. Thermal diffusion effects are then important in a thin region centred round the "dividing streamline" which merges with the inner solution and at the outer edge of which the fluid attains the ambient conditions and so the temperature passes smoothly from T_w near the wall to the ambient temperature T_0.

2. EQUATIONS

If we assume that the convective fluid and the porous medium are isotropic, in thermodynamic equilibrium and have constant physical properties and that the Boussinesq approximation is valid, then with the usual boundary-layer simplifications, the governing equations are, from Wooding [3],

$$u = \frac{g\beta K\rho_0}{\mu}(T-T_0) \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where K is the permeability of the porous medium, α the equivalent thermal diffusivity and ρ_0 the density of the ambient fluid. u and v are the velocities, as given by Darcy's law, in the x and y directions respectively. The boundary conditions are

$$T = T_w,\quad v = \pm V_w \quad \text{on} \quad y = 0 \tag{4}$$

$$T \to T_0,\quad u \to 0 \quad \text{as} \quad y \to \infty \tag{5}$$

where $V_w(>0)$ is the transpiration velocity. The upper sign is taken throughout for the injection of fluid and the lower sign for the withdrawal of fluid.

3. NUMERICAL SOLUTION

From (2) we can define a stream function ψ in the usual way and then make the equations non-dimensional by writing

$$\psi = \frac{\alpha}{V_w}\left(\frac{\rho_0 g\beta\Delta TK}{\mu}\right)F(X,Y),$$

$$T-T_0 = \Delta T\theta(X,Y)$$

$$X = \frac{V_w^2}{\alpha}\left(\frac{\mu}{\rho_0 g\beta\Delta TK}\right)x \quad\text{and}\quad Y = \frac{V_w}{\alpha}y$$

where $\Delta T = |T_w - T_0|$.

Equation (1) then gives $\theta = \partial F/\partial Y$ and, using this, equation (3) becomes

$$\frac{\partial^3 F}{\partial Y^3} = \frac{\partial F}{\partial Y}\frac{\partial^2 F}{\partial Y\partial X}-\frac{\partial F}{\partial X}\frac{\partial^2 F}{\partial Y^2} \tag{6}$$

with boundary conditions

$$\left.\begin{aligned}&\frac{\partial F}{\partial X} = \mp 1,\quad \frac{\partial F}{\partial Y} = 1 \quad\text{on}\quad Y = 0,\\ &\frac{\partial F}{\partial Y}\to 0 \quad\text{as}\quad Y\to\infty.\end{aligned}\right. \tag{7}$$

Near the leading edge the flow will be driven by the buoyancy forces and a further transformation is required to put equation (6) in a form more appropriate for solving from $X = 0$. To do this we write

$$F = \mp\frac{\xi^2}{2}+\xi f(\xi,\eta) \text{ where } \eta = \frac{Y}{\xi} \text{ and } \xi = (2X)^{1/2}.$$

Equation (6) then becomes

$$\frac{\partial^3 f}{\partial\eta^3}+f\frac{\partial^2 f}{\partial\eta^2}\mp\xi\frac{\partial^2 f}{\partial\eta^2} = \xi\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial\xi}-\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}\right) \tag{8}$$

with boundary conditions

$$f = 0,\ \frac{\partial f}{\partial\eta} = 1 \text{ on } \eta = 0,\ \frac{\partial f}{\partial\eta}\to 0 \text{ as } \eta\to\infty. \tag{9}$$

$f(\xi,\eta)$ is then expanded in the form

$$f(\xi,\eta) = f_0(\eta)\mp\xi f_1(\eta)+\xi^2 f_2(\eta)\pm\ldots. \tag{10}$$

The equation for $f_0(\eta)$ is the Blasius equation

$$f_0'''+f_0 f_0'' = 0 \tag{11}$$

with $f_0(0) = 0$, $f_0'(0) = 1$ and $f_0'\to 0$ as $\eta\to\infty$ (primes denote differentiation with respect to η). Equation (11) has arisen previously in a different context and its numerical solution is treated in some detail by Ackroyd [5]. The equations for f_1 and f_2 are linear and can be solved in a straightforward manner. We can define a heat-transfer coefficient Q by

$$Q = -\frac{\alpha}{V_w\Delta T}\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\left(\frac{\partial^2 F}{\partial Y^2}\right)_0,$$

then, from (10), we have for small ξ

$$Q = \xi^{-1}(0.62756\mp 0.44713\xi+0.11942\xi^2\mp\ldots). \tag{12}$$

Equation (8) was solved numerically using essentially the same method as described in [4]. In the case of the withdrawal of fluid the numerical integration started at $\xi = 0$ and proceeded in steps of $\Delta\xi = 0.1$ to $\xi = 1$ (i.e. $X = 0.5$). The profiles thus obtained were used as starting values for a numerical integration of equation (6) which is more appropriate for large X. This then proceeded from $\xi = 1$ until the asymptotic values were attained to within the accuracy of the numerical scheme. For the case of the injection of fluid, equation (8) was integrated from $\xi = 0$ without any change until the asymptotic values were reached. Values of the heat-transfer coefficient Q for the injection and withdrawal of fluid are given in Tables 1 and 2 respectively. Also given in Tables 1 and 2 are the values of Q as calculated from (12) for small ξ, and it can be seen that there is good agreement up to about $\xi = 1$.

4. ASYMPTOTIC EXPANSION – WITHDRAWAL OF FLUID

The fluid in the boundary layer is accelerated by the buoyancy forces resulting from the applied temperature difference, and, if there were no with-

Table 1. Heat transfer Q for the injection of fluid

ξ	X	Numerical solution	Series [from (12)]
0.2	0.02	2.7147	2.7146
0.4	0.08	1.1683	1.1695
0.6	0.18	0.6666	0.6705
0.8	0.32	0.4256	0.4329
1.0	0.50	0.2883	0.2999
1.2	0.72	0.2022	0.2191
1.4	0.98	0.1450	0.1683
1.6	1.28	0.1056	
1.8	1.62	0.0776	
2.0	2.00	0.0573	
2.4	2.88	0.0316	
2.8	3.92	0.0174	
3.2	5.12	0.0095	
3.6	6.48	0.0051	
4.0	8.00	0.0027	
4.4	9.68	0.0014	
4.8	11.52	0.0007	
5.2	13.52	0.0003	
5.6	15.68	0.0002	
6.0	18.00	0.0001	
6.4	20.48	0.0000	

Table 2. Heat transfer Q for the withdrawal of fluid

ξ	X	Numerical solution	Series [from (12)]
0.2	0.02	3.6089	3.6089
0.4	0.08	2.0650	2.0638
0.6	0.18	1.5678	1.5647
0.8	0.32	1.3325	1.3271
1.0	0.50	1.2020	1.1941
	0.70	1.1289	1.1188
	0.90	1.0871	1.0751
	1.10	1.0609	1.0474
	1.30	1.0434	
	1.50	1.0314	
	1.90	1.0169	
	2.50	1.0069	
	3.30	1.0023	
	4.10	1.0008	
	4.90	1.0004	
	5.70	1.0002	
	6.50	1.0001	
	7.30	1.0000	

drawal of fluid through the wall, its thickness would increase like $X^{1/2}$. Withdrawing the fluid removes the warmest fluid and so reduces the acceleration of the fluid in the boundary layer. This has the effect of decreasing the boundary-layer thickness so for large X a balance is reached between the retarding effect of withdrawing the fluid and the accelerating effect of the buoyancy forces, and in the limit as $X \to \infty$ the boundary layer will have constant thickness. This is confirmed by the numerical integration as described in the previous section.

This suggests looking for a solution of (6) in the form

$$F(X, Y) = X + F_0(Y) + F_1(X, Y) \tag{13}$$

where, for large X, F_1 is small compared to F_0. Substituting (13) in (6) and retaining only the largest terms gives an ordinary differential equation for $F_0(Y)$ which has the solution

$$F_0(Y) = 1 - e^{-Y}. \tag{14}$$

The equation for $F_1(X, Y)$ is then, on retaining only the lowest order terms,

$$\frac{\partial^3 F_1}{\partial Y^3} + \frac{\partial^2 F_1}{\partial Y^2} - e^{-Y}\left(\frac{\partial F_1}{\partial X} + \frac{\partial^2 F_1}{\partial X \partial Y}\right) = 0 \tag{15}$$

with boundary conditions

$$F_1 = \frac{\partial F_1}{\partial Y} = 0 \text{ on } Y = 0, \quad \frac{\partial F_1}{\partial Y} \to 0 \text{ as } Y \to \infty. \tag{16}$$

As in [4] we must look for a solution of (15) in the form $F_1(X, Y) = e^{-\gamma X}\phi(Y)$ where $\phi(Y)$ satisfies the eigenvalue problem

$$\phi''' + \phi'' + \gamma e^{-Y}(\phi' + \phi) = 0 \tag{17}$$

with $\phi(0) = \phi'(0) = 0$, $\phi' \to 0$ as $Y \to \infty$ (primes denote differentiation with respect to Y). The first four eigenvalues are found to be $\gamma_1 = 1.4458$, $\gamma_2 = 7.6188$, $\gamma_3 = 18.7226$ and $\gamma_4 = 34.7622$, with corresponding eigensolutions ϕ_1, ϕ_2, ϕ_3 and ϕ_4. Graphs of ϕ'_1, ϕ'_2, ϕ'_3 and ϕ'_4 [normalised so that $\phi''_i(0) = 1$] are given in Fig. 1. The heat-transfer coefficient Q is

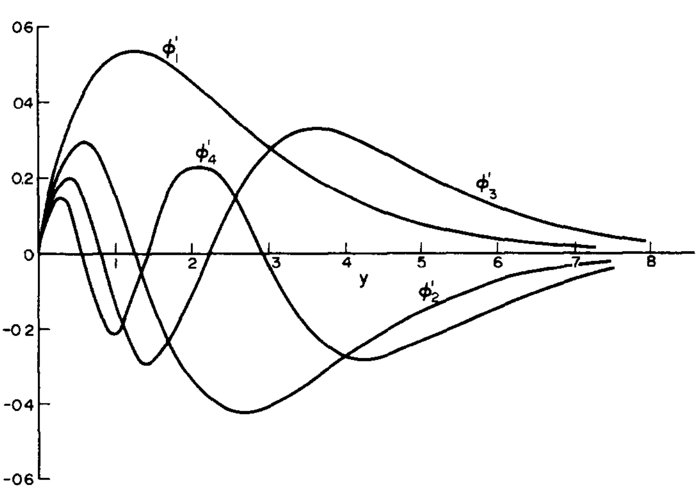

FIG. 1. Eigensolutions ϕ'_1, ϕ'_2, ϕ'_3 and ϕ'_4.

then, for large X,

$$Q = 1+\delta e^{-\gamma X} \tag{18}$$

where δ is a constant which cannot be determined from the asymptotic expansion. A comparison with the numerical solution gives $\delta = 0.26$.

5. ASYMPTOTIC EXPANSION – INJECTION OF FLUID

Non-dimensional temperature profiles θ are plotted in Fig. 2 for various X. From this figure we can see that, for large X, there is a region next to the wall in which $\theta = 1$ and that its thickness is proportional to X. This region is made up of fluid that has been injected through the wall and in it the effects of thermal diffusion are negligible. At the edge of this inner region is a thinner region where thermal diffusion is important and at the outer edge of which the ambient conditions are attained.

gives

$$\frac{\partial^3 G}{\partial \zeta^3}+G\frac{\partial^2 G}{\partial \zeta^2} = \xi\left(\frac{\partial G}{\partial \zeta}\frac{\partial^2 G}{\partial \zeta\,\partial \xi}-\frac{\partial G}{\partial \xi}\frac{\partial^2 G}{\partial \zeta^2}\right). \tag{20}$$

The boundary conditions are that $\partial G/\partial\zeta \to 0$ as $\zeta \to \infty$ and that the solution should merge with the inner solution near the wall, given by $\zeta = -\xi/2$. However, as we are considering a solution as $\xi \to \infty$ this inner boundary condition can be applied as $\zeta \to -\infty$ provided the inner solution is approached with exponentially small error.

$G(\xi,\zeta)$ is expanded as

$$G(\xi,\zeta) = G_0(\zeta)+\xi^{-1}G_1(\zeta)+\xi^{-2}G_2(\zeta)+\ldots \tag{21}$$

$G_0(\zeta)$ satisfies the equation

$$G_0'''+G_0G_0'' = 0 \tag{22}$$

with $G_0' \to 0$ as $\zeta \to \infty$ and $G_0 \sim \zeta$ as $\zeta \to -\infty$ (primes denote differentiation with respect to ζ).

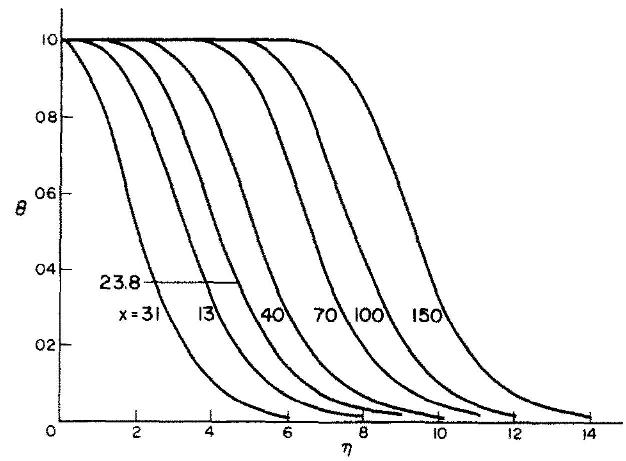

FIG. 2. Non-dimensional temperature profiles θ at various X – injection of fluid.

In the inner region we have

$$F = Y-X, \quad \theta = 1. \tag{19}$$

(19) satisfies the boundary conditions only in $Y = 0$. The streamlines of this inner solution are the straight lines $Y = X-X_0$. X_0 being the point on the wall where the particular streamline emerged. Since for $X < 0$ the fluid is at rest and at the ambient temperature (19) must be confined within the region $0 \leqslant Y \leqslant X$ and for $Y > X$ we must have $F = 0$, $\theta = 0$. The "dividing streamline" $Y = X$ is the one that emerged from the wall at $X = 0$. Since (19) then gives a discontinuity in temperature on $Y = X$ there will be a region centred round the "dividing streamline" where thermal diffusion effects have to be included.

A consideration of the orders of magnitude of the terms in equation (6) shows that the outer region has a thickness of $O(X^{1/2})$. This suggests putting $F = \xi G(\xi,\zeta)$ where $\zeta = \xi^{-1}(Y-\xi^2/2)$ and $\xi = (2X)^{1/2}$. So the "dividing streamline" $Y = X$ becomes $\zeta = 0$ and the inner solution is $G = \zeta$. Equation (6) then

Values of G_0' are given in Table 3. To check that G_0 approaches the inner boundary condition with an exponentially small error put $G_0 = \zeta+H_0(\zeta)$ then, as $\zeta \to -\infty$, equation (22) becomes $H_0'''+\zeta H_0'' = 0$ which has the solution

$$H_0' = C_0\,\mathrm{erfc}[\zeta/(2)^{1/2}] \sim \frac{C_0}{(\pi)^{1/2}}\frac{\exp(-\zeta^2/2)}{\zeta}.$$

The equations for G_1 and G_2 can be solved in terms of G_0, namely, $G_1 = B_0(1-G_0')$ and $G_2 = \frac{1}{2}B_0^2G_0''+B_1(\zeta G_0'-G_0)$. B_0 and B_1 are undetermined constants which arise from the asymptotic nature of the solution in the sense described by Stewartson [6].

6. CONCLUSION

For a constant wall temperature Cheng [2] found that he needed a transpiration velocity which decreased like $x^{-1/2}$ to obtain a similarity solution, and then the boundary-layer thickness increased like $x^{1/2}$ for both the injection and withdrawal of fluid. When we have the more realistic condition of a

Table 3.

ζ	G_0'
−4.0	1.0000
−3.5	0.9999
−3.0	0.9992
−2.5	0.9962
−2.0	0.9862
−1.5	0.9592
−1.0	0.9025
−0.8	0.8692
−0.6	0.8295
−0.4	0.7839
−0.2	0.7330
0.0	0.6910
0.2	0.6357
0.4	0.5791
0.6	0.5227
0.8	0.4677
1.0	0.4150
1.5	0.2982
2.0	0.2063
2.5	0.1385
3.0	0.0905
3.5	0.0575
4.0	0.0353
4.5	0.0205
5.0	0.0107
5.5	0.0042
6.0	0.0000

constant transpiration velocity we find that in the case of withdrawal of fluid the boundary layer remains very thin and quickly settles down to one of constant thickness. Using the same data as Cheng [2], namely $\rho_0 = 9.2 \times 10^2\,\mathrm{kg\,m^{-3}}$, $\alpha = 6.3 \times 10^{-7}\,\mathrm{m^2\,s^{-1}}$, $\beta = 2.8 \times 10^{-4}/\mathrm{K}$, $g = 9.8\,\mathrm{m^2\,s^{-1}}$, $\Delta T = 75°\mathrm{K}$, $K = 10^{-10}\,\mathrm{m^2}$, $\mu = 6.8 \times 10^{-4}\,\mathrm{kg\,m^{-1}\,s^{-1}}$ and a transpiration velocity $V_w = 2 \times 10^{-6}\,\mathrm{m\,s^{-1}}$ we find that the boundary layer has a constant thickness of about 2 m at approximately 25 m from the leading edge. When fluid is discharged through the wall there is a region of thickness proportional to x near the wall at the same temperature as the discharged fluid. Cheng [2] found, using the above data, that his similarity solution gave a boundary-layer thickness at 500 m from the leading edge of about 30 m. In the present case, we find, at the same point, an inner region of thickness about 50 m and an outer region of thickness about 30 m. So that the effect on the ambient conditions of the discharge of water through the wall is spread over a wider region.

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COUCHES LIMITES DE CONVECTION NATURELLE DANS UN MILIEU POREUX SATURE AVEC FLUX MASSIQUE LATERAL

Résumé—On considère les effets d'un flux massique latéral uniforme sur la couche limite de convection naturelle sur une paroi verticale dans un milieu poreux saturé. On obtient une série valable près du bord d'attaque et elle est étendue par une solution numérique des équations complètes. Des développements asymptotiques, valables à de grandes distances le long de la plaque, sont établis à la fois dans le cas de sortie et d'injection du fluide. Dans le premier cas la couche limite a une épaisseur constante tandis que dans le dernier, il y a une région de température constante près de la paroi à cause du fluide qui a été injecté à travers la paroi, avec une région externe où la diffusion thermique est importante.

GRENZSCHICHTEN BEI FREIER KONVEKTION IN EINEM GESÄTTIGTEN PORÖSEN MEDIUM MIT SEITLICHEM MASSENSTROM

Zusammenfassung—Die Einflüsse eines gleichmäßigen seitlichen Massenstroms auf die Grenzschicht bei freier Konvektion an einer vertikalen Wand in einem gesättigten porösen Medium werden untersucht. Ein an der Anströmkante gültiger Reihenausdruck wird abgeleitet und der Lösungsbereich durch eine numerische Lösung des vollständigen Gleichungssystems ausgedehnt. Asymptotische Reihenentwicklungen, gültig bei großen Entfernungen entlang der Platte, werden für den Fall der Absaugung und der Einspritzung des Fluids abgeleitet. Im ersten Fall hat die Grenzschicht konstante Dicke, während im letzteren Fall ein Gebiet konstanter Temperatur entsprechend dem durch die Wand eingespritzten Fluid vorhanden ist, an das sich eine äußere Region anschließt, in der Wärmeleitung vorherrscht.

СВОБОДНОКОНВЕКТИВНЫЕ ПОГРАНИЧНЫЕ СЛОИ В НАСЫЩЕННОЙ ПОРИСТОЙ СРЕДЕ ПРИ НАЛИЧИИ ПОПЕРЕЧНОГО ПОТОКА МАССЫ

Аннотация — Рассматривается влияние постоянного поперечного потока массы на свободноконвективный пограничный слой у вертикальной стенки в насыщенной пористой среде. Для окрестности передней кромки выведен ряд, используемый при численном решении полных уравнений. Асимптотические разложения, справедливые для больших расстоянии вдоль пластины, получены для обоих случаев — отсоса и вдува жидкости. В первом случае имеет место пограничный слой постоянной толщины, во втором непосредственно у стенки инжектируемая жидкость имеет постоянную температуру, а во внешней области слоя существенной является теплопроводность.